MATH4060 Tutorial 3

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Problem 1 (Exercise 2)**.** *Find the order of growth of the following entire functions:*

(a) $p(z)$ *where* p *is a polynomial.*

(b) $f(z) = e^{bz^n}$ *for* $b \neq 0$.

(c) $g(z) = e^{e^z}$.

(a) Assume that $p(z) = a_n z^n + \cdots + a_0$ ($a_n \neq 0$). We claim that the order of growth is 0. Indeed, for any $\rho > 0$, we have

$$\frac{|p(z)|}{e^{|z|^\rho}} \leq \frac{a_n|z|^n + \cdots + |a_0|}{e^{|z|^\rho}} \to 0$$

as $|z| \to \infty$, so $|p(z)| \leq A_\rho e^{|z|^\rho}$ for some $A_\rho > 0$. Taking infimum implies that $\rho_p = 0$.

(b) It is clear that $|e^{bz^n}| \leq e^{|b||z|^n}$, so $\rho_f \leq n$. On the other hand, write $b = r_0 e^{i\theta_0}$, where $r_0 > 0$, and consider $z = re^{-i\theta_0/n}$ for any $r > 0$, so that

$$|f(z)| = |e^{bz^n}| = e^{r_0 r^n}.$$

If $|f(z)| \leq Ae^{B|z|^\rho}$, taking $r \to \infty$ in the above shows that this is only possible when $\rho \geq n$. So $\rho_f = n$.

(c) For any $\rho > 0$, we have

$$\lim_{x\to\infty} \frac{e^{e^x}}{e^{Bx^\rho}} = \lim_{x\to\infty} e^{e^x - Bx^\rho} = \infty.$$

So $\rho_g = \infty$.

Problem 2 (Exercise 10a)**.** *Find the Hadamard product for* $f(z) = e^z - 1$.

Method 1: It is easy to see that $\rho_f = 1$ and that the zeros of f are $a_n = 2\pi i n$, $n \in \mathbb{Z}$ (each has order 1). Hadamard's theorem implies

$$\begin{aligned} f(z) &= e^{az+b} z \prod_{n=1}^\infty E_1(z/a_n)E_1(z/a_{-n}) \\ &= e^{az+b} z \prod_{n=1}^\infty (1 - z/a_n)e^{z/a_n}(1 + z/a_n)e^{-z/a_n} \\ &= e^{az+b} z \prod_{n=1}^\infty (1 + z^2/4\pi^2 n^2). \end{aligned}$$

We want to determine a and b. Note that if z_0 is a zero of an entire function f and $f(z) = (z - z_0)g(z)$, then $f'(z_0) = g(z_0)$ and $f''(z_0) = 2g'(z_0)$. Here, set $z_0 = 0$ and $f(z) = e^z - 1$, so $f'(0) = f''(0) = 1$; write $g(z) = e^{az+b}h(z) = e^{az+b}\prod_{n=1}^\infty h_n(z)$, where $h_n(z) = 1 + z^2/4\pi^2n^2$. Since $1 = f'(0) = g(0) = e^b$, we may take $b = 0$. On the other hand, we compute that

$$\frac{g'(z)}{g(z)} = \frac{ae^{az}h(z) + e^{az}h'(z)}{e^{az}h(z)} = a + \frac{h'(z)}{h(z)} = a + \sum_{n=1}^\infty \frac{h_n'(z)}{h_n(z)},$$

by Proposition 3.2 (ii). Since $g(0) = 1$ and $h_n'(0) = 0$, we have $g'(0) = a$. Then $1 = f''(0) = 2g'(0)$ implies that $a = 1/2$. (In fact, for this example, one could also deduce that $a = 1/2$ from

$$e^{z/2} - e^{-z/2} = e^{(a-1/2)z}\left[z\prod_{n=1}^{\infty}(1 + z^2/4\pi^2 n^2)\right]$$

by observing that the LHS and the bracketed term are both odd functions.)

Method 2: Using the Hadamard product for sine, i.e.

$$\sin z = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right) = z\prod_{n\neq 0} E_1\left(\frac{z}{\pi n}\right),$$

we have

$$\begin{aligned} e^z - 1 &= e^{z/2}(e^{z/2} - e^{-z/2}) = 2ie^{z/2}\sin(-iz/2) \\ &= 2ie^{z/2}(-iz/2)\prod_{n\neq 0} E_1(z/(2\pi in)) \\ &= e^{z/2}z\prod_{n\neq 0} E_1(z/(2\pi in)). \end{aligned}$$

Problem 3 (Exercise 17b, Pringsheim interpolation formula)**.** *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of distinct complex numbers such that $a_0 = 0$ and $\lim_{k\to\infty}|a_k| = \infty$. Given complex numbers $\{b_k\}_{k=0}^{\infty}$, show that there exists an entire function F that satisfies $F(a_k) = b_k$ for all k.*

Let $E(z)$ denote a Weierstrass product associated with $\{a_k\}$. For $k \geq 0$, define the interpolation factors

$$\phi_k(z) = \frac{E(z)}{E'(a_k)(z - a_k)}.$$

We claim that ϕ_k is well-defined, entire and satisfies $\phi_k(a_j) = \delta_{kj}$ (using the Kronecker delta notation). Indeed, since the a_k's are distinct, $E(z)$ has simple zeros at the a_k's, and if $E(z) = (z - a_k)g_k(z)$, $E'(a_k) = g_k(a_k) \neq 0$. Hence, ϕ_k is well-defined, entire, and satisfies $\phi_k(a_k) = 1$. If $k \neq j$, then $E(a_j) = 0$ and $a_j - a_k \neq 0$, so $\phi_k(a_j) = 0$.

Next, define

$$F(z) = b_0\phi_0(z) + \sum_{k=1}^{\infty} b_k\phi_k(z)\left(\frac{z}{a_k}\right)^{m_k},$$

for integers $m_k \geq 1$ specified as follow: let $c_k = |b_k/E'(a_k)|$ and choose $m_k \geq 1$ such that

$$\frac{c_k}{2^{m_k - 1}} \leq \frac{1}{2^k}.$$

Note that $F(a_k) = b_k$ for all k by construction. To prove that $F(z)$ is entire, it suffices to show that the series converges uniformly in every closed disc $\overline{D}_R(0)$. Fix $R > 0$ and let $|E(z)| \leq B_R$ on $|z| \leq R$ for some $B_R > 0$. Consider[1] $K \in \mathbb{N}$ such that $|a_j| \geq \max\{2R, B_R\}$ for $j \geq K$. For any $k \geq K$ and $|z| \leq R$, we have

$$\begin{aligned} \left|\sum_{j=k}^{\infty} b_j\phi_j(z)\left(\frac{z}{a_j}\right)^{m_j}\right| &\leq \sum_{j=k}^{\infty}\frac{B_R|z|}{(|a_j| - |z|)|a_j|}c_j\left|\frac{z}{a_j}\right|^{m_j - 1} \\ &\leq \sum_{j=k}^{\infty}\frac{B_R R}{RB_R}\frac{c_j}{2^{m_{j-1}}} \leq \sum_{j=k}^{\infty}\frac{1}{2^j} = \frac{1}{2^{k-1}} \to 0 \end{aligned}$$

uniformly in z, as $k \to \infty$.

[1] I slightly simplified the proof that appeared in the tutorial.

Problem 4 (Exercise 11). *Show that if f is an entire function of finite order that omits two values, then f is a constant.*

Suppose f omits the value $a \in \mathbb{C}$. Hadamard's theorem implies that the nowhere vanishing function $f(z) - a = e^{p(z)}$ for some polynomial $p(z)$. If p is not a constant, the fundamental theorem of algebra implies that $\text{Image}(p) = \mathbb{C}$. Since the exponential function assumes every nonzero value, $f(z) = e^{p(z)} + a$ can only omit the value a in this case.